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The stability of finite amplitude cellular convection and its relation to an extremum principle

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The stability of cellular convection flow in a layer heated from below is discussed for Rayleigh number R close to the critical value R_c. It is shown that in this region the stable stationary solution is determined by a minimum of the integral

$$\int_0^{H_0} R(H)\,dH,$$

where $R(H)$ is a functional of arbitrary convective velocity fields which satisfy the boundary conditions. For the stationary solutions $R(H)$ is equal to the Rayleigh number. H_0 is a given value of the convective heat transport. In a second part of the paper explicit results are derived for the convection problem with deviations from the Boussinesq approximation owing to the temperature dependence of the material properties.

1. Introduction

The streamlines of a stationary flow usually reflect in their shape the conditions imposed from outside. Cellular motion, however, has the peculiar property that its streamlines are confined to domains, so-called cells, which are not determined by the exterior conditions.

From the formal point of view, cellular motion can be characterized as a stationary solution of the equations of motion, which is not uniquely determined by the given boundary conditions. This indeterminacy is not caused by the lack of imposed conditions; rather it can be described by the existence of a bifurcation point of the stationary solution in dependence on a parameter of the problem. While below a certain critical value of the parameter the boundary conditions are sufficient to determine the stationary solution uniquely, two or more stationary solutions are possible above that value. In this case, the criterion of stability against disturbances distinguishes the physically realized solutions among the class of possible stationary solutions. It may be that this criterion, which usually is restricted to disturbances of infinitesimal amplitude, is not sufficient to determine the stable solution uniquely. In general, however, it distinguishes a small class of physically realizable solutions out of a large class of possible solutions.

The unusual properties of cellular motion have attracted many investigators. The most famous example of cellular motion has become the convection in a horizontal layer heated from below. In this problem the relevant parameter is the temperature difference between the upper and lower boundaries of the layer, which in the dimensionless description is represented by the Rayleigh number. For the extended literature on this problem, we refer to Chandrasekhar (1961) and to a recent review article by Segel (1966).

In one of the earliest papers on the nonlinear aspects of the convection problem, Malkus & Veronis (1958) used the suggestive hypothesis that the property of stability of a stationary cellular motion is correlated to the property of maximum convective heat transport at a given Rayleigh number. When a scalar function does exist which assumes an extremal value in the case of the physically realized solution, the stability analysis can be replaced by variational techniques. Hence, the problem of the existence of such a function is of considerable importance. In the first part of this paper we investigate the conditions under which the property of stability of a stationary solution can be derived from the existence of a minimum of a scalar function. We will show that such a function exists for convection flows of sufficiently small amplitudes. It turns out that this minimum principle can be identified in some cases with the principle of maximum convective heat transport.

To prove the minimum principle we will restrict ourselves to a special case of the convection problem as a model. In this case, which is introduced in §2, it is assumed that the Boussinesq approximation is valid and that the boundary conditions are symmetric. In contrast to the usual treatment, we shall admit a nonlinear dependence of the density on the temperature. We shall, however, treat the deviation from the linear dependence as a small perturbation. The analysis of the problem is based on the expansion of the variables in powers of the convection amplitude. This method of solution was introduced for the solution of the nonlinear stationary equations by Malkus & Veronis (1958) and was applied to the stability analysis by Schlüter, Lortz & Busse (1965). The present work can be considered an extension of the latter paper, to which we will refer hereafter as I. The proof of the minimum principle is contained in §§3 and 4. In §5 we complete the presentation of the principle by discussing its general implications.

The second part of the paper, §§6 to 9, is concerned with the question of how the stability of the stationary solutions, as described in I, is altered by deviations from the Boussinesq approximation. We shall assume that all material properties are temperature-dependent and treat the deviations from the Boussinesq approximation as small perturbations in analogy to the nonlinear terms. Since the equations for this problem become rather extended, the discussion will be based essentially on the special case introduced in §2, which shows all qualitative features. The solution of this problem, as well as a more restricted formulation of the minimum principle, is part of the author's dissertation (1962) to which we shall refer as II for certain details.

2. The formulation of the problem

Convective motions in a fluid layer heated from below are described by the continuity equation, the Navier–Stokes equations of motion and the heat equation. We will use these equations in the Boussinesq approximation in which all material properties are assumed to be constant with the exception of the density, whose temperature dependence

$$\rho = \rho_0[1-\alpha(T-T_0)-\beta(T-T_0)^2+\ldots] \tag{2.1}$$

is taken into account in the gravity term only. We assume that the reference temperature T_0 in the expansion (2.1) is given by the temperature $(T_1+T_2)/2$ of the static state in the middle plane of the layer where T_1 and T_2 are the given temperatures at the lower and upper boundaries of the layer. In order to write the equations in a dimensionless form, we introduce the thickness d of the layer as the length scale and $d^2\kappa^{-1}$ as the time scale, where κ is the thermometric conductivity. As the scale for the temperature we introduce the difference T_1-T_2 divided by the Rayleigh number R. Thus the momentum equation and the heat equation for the velocity vector u_i and the deviation of temperature from the static state θ have the following form:

$$\left.\begin{aligned} \Delta u_i+\lambda_i\left(\theta-2\gamma\lambda_j x_j\theta+\frac{\gamma\theta^2}{R}\right)-\partial_i p &= Pr^{-1}\left(u_j\partial_j u_i+\frac{\partial}{\partial t}u_i\right),\\ \Delta\theta+R\lambda_j u_j &= u_j\partial_j\theta+\frac{\partial}{\partial t}\theta,\\ \partial_j u_j &= 0,\end{aligned}\right\} \tag{2.2}$$

where the summation convention and the notation $\partial_j \equiv \partial/\partial x_j$ have been used. We assume a system of Cartesian co-ordinates with x, y in the horizontal directions and z in the vertical direction opposite to the direction of the gravity force. In this system the unit vector $\boldsymbol{\lambda}$ has the components $(0, 0, 1)$. Unlike the conventional use of the Boussinesq approximation we take into account the quadratic term in the temperature expansion (2.1) of the density. We will consider, however,

$$\gamma = \beta(T_1-T_2)/\alpha \tag{2.3}$$

as a small parameter. The other dimensionless parameters of the problem are the Rayleigh number

$$R = \alpha g d^3(T_1-T_2)/\nu\kappa$$

and the Prandtl number $Pr = \nu/\kappa$, where ν is the kinematic viscosity. To simplify the discussion in the following sections we will assume the limit of infinite Prandtl number, in which the right side of the first equation in (2.2) can be neglected. In a later part of this paper it will be shown that all conclusions remain valid in the case of finite Prandtl number.

Two cases are representative for the various possibilities of boundary conditions for the velocity vector. At the rigid boundary the velocity vector u_i has to vanish, while at the free boundary only the normal component of the velocity is vanishing, but no tangential stress can be supported. Since we have assumed that constant temperatures are prescribed at the boundaries and because we will consider symmetric cases only, the boundary conditions of the problem are given by

$$\left.\begin{aligned} u &= 0, \quad \theta = 0 \text{ for rigid boundaries at } z = \pm\tfrac{1}{2},\\ u_z &= \partial_z u_{x,y} = \theta = 0 \text{ for free boundaries at } z = \pm\tfrac{1}{2}.\end{aligned}\right\} \tag{2.4}$$

An important consequence of (2.2) in the limit of infinite Prandtl number is that the vertical component of the curl of the velocity vanishes since the corresponding part of the equations of motion

$$\Delta(\partial_x u_y - \partial_y u_x) = 0, \tag{2.5}$$

together with the boundary conditions, admits only the vanishing solution. An arbitrary velocity field with vanishing divergence and vanishing vertical component of the vorticity can be derived from a scalar function v,

$$u_i = \delta_i v, \tag{2.6}$$

with the operator

$$\delta_i = \partial_i \partial_j \lambda_j - \lambda_i \partial_j \partial_j = (\partial_x \partial_z, \partial_y \partial_z, -\Delta_2), \tag{2.7}$$

where, for abbreviation, $\Delta_2 = \partial^2_{xx} + \partial^2_{yy}$ has been introduced.

Eliminating the pressure term we reduce (2.2) to the following equations for v and θ:

$$\left.\begin{aligned} \Delta\Delta v - \theta &= -\gamma\left(2z\theta - \frac{\theta\theta}{R}\right), \\ \Delta\theta - R\Delta_2 v &= \delta_j v \partial_j \theta + \frac{\partial}{\partial t}\theta. \end{aligned}\right\} \tag{2.8}$$

Assuming the stationary case with infinitesimal amplitude ϵ of v and $\gamma = 0$, we can neglect the right side of (2.8) and obtain for $v = \epsilon v^{(10)}$ the linear equation

$$(\Delta\Delta\Delta - R\Delta_2)\, v^{(10)} = 0. \tag{2.9}$$

The properties of this equation together with various homogeneous boundary conditions are well known; we refer to Chandrasekhar (1961). The variables can be separated in this case,

$$\Delta_2 v^{(10)} = -a^2 v^{(10)}, \tag{2.10}$$

and stationary solutions exist depending on the boundary conditions for certain values of $R = R(a)$. The minimum value of $R(a)$ which we will name $R^{(00)}$ is called the critical value because it determines the point at which the static layer becomes unstable.

Starting with the solution $\epsilon v^{(10)}$ at $R = R^{(00)}$ we assume the following expansion for the stationary solution of the problem (2.8) for small values of the parameters ϵ and γ:

$$\left.\begin{aligned} v &= \sum_{\substack{\mu=1 \\ \nu=0}}^{\infty} \epsilon^{\mu}\gamma^{\nu} v^{(\mu\nu)}, \\ \theta &= \sum_{\substack{\mu=1 \\ \nu=0}}^{\infty} \epsilon^{\mu}\gamma^{\nu} \theta^{(\mu\nu)}, \\ R &= \sum_{\substack{\mu=0 \\ \nu=0}}^{\infty} \epsilon^{\mu}\gamma^{\nu} R^{(\mu\nu)}. \end{aligned}\right\} \tag{2.11}$$

We assume that ϵ is a positive parameter, while γ may have either sign. The idea of the method of solution is to solve (2.8) after introducing expansion (2.11) for each power of ϵ and γ separately. Hence, instead of the nonlinear problem (2.8), a set of linear inhomogeneous equations has to be solved starting with the homogeneous equation (2.9).

According to the separation relation (2.10) the solution $v^{(10)}$ can be written in the following general form:

$$v^{(10)} = \sum_n c_n v_n, \tag{2.12}$$

where

$$v_n = \exp\{i\mathbf{k}_n.\mathbf{r}\}f(z) \tag{2.13}$$

represents the complete system of solutions of (2.9) for $R = R^{(00)}$. We assume that the summation in (2.12) runs through all integers different from zero. The class of vectors $\mathbf{k}_n$ is defined by

$$\mathbf{k}.\boldsymbol{\lambda} = 0, \quad \mathbf{k}.\mathbf{k} = a_c^2. \tag{2.14}$$

a_c is the value of the wave-number a which corresponds to the critical Rayleigh number.

In order for solution (2.12) to be real, we assume $\mathbf{k}_{-n} = -\mathbf{k}_n$ and $c_{-n} = \bar{c}_n$. As a normalization condition we introduce

$$\sum_n c_n \bar{c}_n = 1. \tag{2.15}$$

Because of the separation of variables the solutions of the adjoint problem to (2.9) are represented by

$$v_n^* = \exp\{-i\mathbf{k}_n.\mathbf{r}\}f^*(z). \tag{2.16}$$

The functions $f(z)$ and $f^*(z)$ are determined by (2.9) together with the corresponding boundary conditions. For the following considerations, however, we do not have to use their explicit form. It is convenient to normalize the solution $v^{(10)}$ by assuming

$$a_c^2\langle v^{(10)}\Delta\Delta v^{(10)}\rangle = 1, \tag{2.17}$$

where the brackets indicate the average over the fluid layer. By this definition ϵ^2 gives the first approximation to the convective heat transport $H = \langle u_z \theta\rangle$.

In the next section it will be shown that those solutions among the general class (2.12) which correspond to solutions of the nonlinear equations (2.8) in the limit of small γ and ϵ are distinguished by the fact that a certain function of the coefficients c_n assumes an extremum, or more exactly, a stationary value. In §4 we shall prove that the stability of the solution depends upon whether or not the stationary value is a minimum.

3. The stationary solution

After introducing the expansion (2.11) into (2.8) we consider terms belonging to different powers in γ and ϵ separately. Eliminating θ from the equations in the same way as has been done for the linear part of (2.8) ,we obtain

$$\begin{aligned}
(\Delta^3 - R^{(00)}\Delta_2)(\epsilon v^{(20)} + \gamma v^{(11)}) &= \epsilon(\delta_j v^{(10)}\partial_j \Delta^2 v^{(10)} + R^{(10)}\Delta_2 v^{(10)}) \\
&\quad - \gamma(2\Delta z \Delta^2 v^{(10)} - R^{(01)}\Delta_2 v^{(10)}), \\
(\Delta^3 - R^{(00)}\Delta_2)(\epsilon^2 v^{(30)} + \epsilon\gamma v^{(21)} + \gamma^2 v^{(12)}) &= \epsilon^2(\delta_j v^{(10)}\partial_j \Delta^2 v^{(20)} + \delta_j v^{(20)}\partial_j \Delta^2 v^{(10)} \\
&\quad + R^{(10)}\Delta_2 v^{(20)} + R^{(20)}\Delta_2 v^{(10)}) - \epsilon\gamma(2\Delta z\Delta^2 v^{(20)} - \delta_j v^{(10)}\partial_j(\Delta^2 v^{(11)} + 2z\Delta^2 v^{(10)}) \\
&\quad - \delta_j v^{(11)}\partial_j \Delta^2 v^{(10)} - (1/R)\Delta(\Delta^2 v^{(10)})^2 - R^{(10)}\Delta_2 v^{(11)} - R^{(01)}\Delta_2 v^{(20)} - R^{(11)}\Delta_2 v^{(10)}) \\
&\quad - \gamma^2(2\Delta z\Delta^2 v^{(11)} + 4\Delta z^2\Delta^2 v^{(10)} - R^{(01)}\Delta_2 v^{(11)} - R^{(02)}\Delta_2 v^{(10)}), \; \dots \; \dots \; \dots\dots
\end{aligned} \tag{3.1}$$

This set of linear inhomogeneous equations can be solved sequentially. To exclude the additional arbitrary solution of the homogeneous problem, we choose the normalization condition

$$\left\langle \sum_n c_n v_n^* \Delta_2 v^{(\mu\nu)} \right\rangle = -\delta_{\mu 1}\delta_{\nu 0} \tag{3.2}$$

As the necessary and sufficient condition for the solvability of a linear inhomogeneous problem, the inhomogeneity has to be orthogonal to all solutions of the adjoint homogeneous problem. Thus the right sides of (3.1) multiplied by the functions (2.16) and averaged over the fluid layer have to vanish. Since $f(z)$ as well as $f^*(z)$ are symmetric functions in z, the solvability condition in the case of the first equation (3.1) is satisfied when

$$R^{(10)} = R^{(01)} = 0. \tag{3.3}$$

The solutions $v^{(20)}$ and $v^{(11)}$ can be written in the form

$$v^{(20)} = \sum_{nm} c_n c_m \exp\{i(\mathbf{k}_n + \mathbf{k}_m) . \mathbf{r}\} F(\mathbf{k}_n . \mathbf{k}_m, z), \tag{3.4}$$

$$v^{(11)} = \sum_n c_n \exp\{i\mathbf{k}_n . \mathbf{r}\} G(z), \tag{3.5}$$

since only the scalar products of $\mathbf{k}$-vectors appear on the right side of (3.1).

We introduce the solutions (3.4, 5) on the right side of the second equation in (3.1). By multiplying it with v_l^* and averaging, we arrive at a set of equations with running index l. Since the averaging integral is extended over terms of the horizontal dependence

$$\exp\{i(-\mathbf{k}_l + \mathbf{k}_n + \ldots) . \mathbf{r}\}$$

with two, three or four $\mathbf{k}$-vectors in the exponent, the set of equations is given by

$$0 = \epsilon^2\left(\sum_{k,n,m} A(\mathbf{k}_n . \mathbf{k}_m, \mathbf{k}_m . \mathbf{k}_k, \mathbf{k}_k . \mathbf{k}_n) c_m c_n c_k \delta(-\mathbf{k}_l + \mathbf{k}_n + \mathbf{k}_m + \mathbf{k}_k) - R^{(20)} c_l\right) + \epsilon\gamma\left(\sum_{n,m} B(\mathbf{k}_n . \mathbf{k}_m) c_m c_n \delta(-\mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n) - R^{(11)} c_l\right) + \gamma^2 (D - R^{(02)}) c_l. \tag{3.6}$$

Due to the form of the operators on the right-hand side of (3.1) A and B are functions of the scalar products between the $\mathbf{k}$-vectors only, and D is a constant. $B(\mathbf{k}_n . \mathbf{k}_m)$ can also be replaced by a constant $B_0 = B(-\frac{1}{2}a_c^2)$, since the argument of the δ-function vanishes only when the three $\mathbf{k}$-vectors form an equilateral triangle. With respect to two of the $\mathbf{k}$-vectors, say $\mathbf{k}_n, \mathbf{k}_m$, A is a symmetric function because the solution $v^{(20)}$ has this symmetry. With the definitions

$$\left.\begin{aligned} \phi_{ln} &\equiv \frac{1}{a_c^2} \mathbf{k}_l . \mathbf{k}_n, \\ L(\phi_{ln}) &\equiv A(\mathbf{k}_l . \mathbf{k}_n, -a_c^2, -\mathbf{k}_l . \mathbf{k}_n) = A(\mathbf{k}_l . \mathbf{k}_n, -\mathbf{k}_l . \mathbf{k}_n, -a_c^2), \\ L_1(\phi_{ln}) &\equiv A(-a_c^2, -\mathbf{k}_l . \mathbf{k}_n, \mathbf{k}_l . \mathbf{k}_n), \end{aligned}\right\} \tag{3.7}$$

the set of equations represented by (3.6) can be rewritten in the form

$$0 = \epsilon^2 \sum_n \{L(\phi_{ln})(2 - 2\delta_{ln} - \delta_{-ln}) + L_1(\phi_{ln})\} c_n c_{-n} c_l + \epsilon\gamma \sum_{n,m} B_0 c_m c_n \delta(\mathbf{k}_m + \mathbf{k}_n - \mathbf{k}_l) + (\gamma^2 D - \epsilon^2 R^{(20)} - \epsilon\gamma R^{(11)} - \gamma^2 R^{(02)}) c_l. \tag{3.8}$$

In this form the solvability condition is equivalent to the condition for a stationary value of the following function E of the coefficients c_n subjected to the side condition (2.15):

$$E(\ldots, c_{-1}, c_1, \ldots) \equiv \tfrac{1}{4}\epsilon^2 \sum_{n,l} \{L(\phi_{ln})(2-2\delta_{ln}-\delta_{-ln}) + L_1(\phi_{ln})\} c_n c_{-n} c_l c_{-l}$$
$$+ (\tfrac{1}{3}\epsilon\gamma) B_0 \sum_{n,m,l} c_l c_m c_n \delta(\mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n) + \tfrac{1}{2}\gamma^2 \sum_l D c_l c_{-l}. \quad (3.9)$$

The comparison of the equations

$$0 = \frac{\partial}{\partial c_{-l}} E' \equiv \frac{\partial}{\partial c_{-l}} \left(E + \lambda \left(\sum_n c_n c_{-n} - 1 \right) \right) \quad (3.10)$$

with equations (3.8) proves the equivalence, identifying

$$\lambda = -\epsilon^2 R^{(20)} - \epsilon\gamma R^{(11)} - \gamma^2 R^{(02)}. \quad (3.11)$$

The equivalence is the basis for our conclusion: *among the class* (2.12) *of solutions, only those are possible solutions of the equations* (2.8) *in the limit of small* ϵ *and* γ, *for which the function* E *assumes a stationary value under the side condition* (2.15).

A class of solutions which satisfy the solvability condition (3.8) is given by

$$c_1 c_{-1} = \ldots = c_N c_{-N} = 1/(2N) \quad (3.12)$$

and corresponding $\mathbf{k}$-vectors $\mathbf{k}_{-N}, \ldots, \mathbf{k}_N$ which have the property that the scalar product between any one of the $\mathbf{k}$-vectors and its two neighbouring $\mathbf{k}$-vectors assumes the constant values α and β. We will call solutions of this form 'semi-regular', leaving the term 'regular' for the special case $\alpha = \beta$. That semi-regular solutions satisfy the solvability condition (3.8) can be seen easily if we disregard solutions for which any scalar product $\mathbf{k}_n \cdot \mathbf{k}_m$ is equal to $\frac{1}{2}a_c^2$. Under this condition the second term in the set of equations (3.8) disappears, and all equations become identical after division by c_l. In §7 we shall return to the case when an angle of 60° occurs between two $\mathbf{k}$-vectors.

Since the example (3.12) represents an infinite class of possible stationary solutions, a stability analysis is necessary to distinguish the physically realized solution. In the next section we will show that the stable solution is characterized by the fact that the stationary value of the function E is a minimum.

4. The stability analysis

Infinitesimal disturbances $\tilde{v}$, $\tilde{\theta}$ superposed on the stationary solution v, θ are governed by the equations

$$\left.\begin{aligned} \Delta\Delta\tilde{v} - \tilde{\theta} &= -2\gamma\left(z\tilde{\theta} - \frac{\theta\tilde{\theta}}{R}\right), \\ \Delta\tilde{\theta} - R\Delta_2\tilde{v} &= \delta_j \tilde{v}\partial_j\theta + \delta_j v \partial_j \tilde{\theta} + \sigma\tilde{\theta}. \end{aligned}\right\} \quad (4.1)$$

We have replaced the derivative of $\tilde{\theta}$ with respect to the time by $\sigma\tilde{\theta}$ because an exponential time dependence can be assumed. Because the stationary solution is given in the form of the expansion (2.11) we introduce analogous expansions for $\tilde{v}$, $\tilde{\theta}$ and σ:

$$\left.\begin{aligned} \tilde{v} &= \sum_{\substack{\mu=1\\ \nu=0}} \epsilon^{\mu-1}\gamma^{\nu}\tilde{v}^{(\mu\nu)},\\ \tilde{\theta} &= \sum_{\substack{\mu=1\\ \nu=0}} \epsilon^{\mu-1}\gamma^{\nu}\tilde{\theta}^{(\mu\nu)},\\ \sigma &= \sum_{\substack{\mu=0\\ \nu=0}} \epsilon^{\mu}\gamma^{\nu}\sigma^{(\mu\nu)}. \end{aligned}\right\} \tag{4.2}$$

At lowest order, (4.1) admits only solutions with $\sigma^{(00)} \leqslant 0$ according to the linear theory. In the case $\sigma^{(00)} = 0$, the equation for $\tilde{v}^{(10)}$ becomes identical with (2.9) for $v^{(10)}$ with $R = R^{(00)}$ and the solution can be written in the form

$$\tilde{v}^{(10)} = \sum_n \tilde{c}_n v_n. \tag{4.3}$$

In the appendix we will show that it is sufficient to restrict the stability analysis to disturbances of this form. The consideration of disturbances with small negative values $\sigma^{(00)}$ will not alter the conclusion.

At higher orders we obtain from the system (4.1) a hierarchy of equations analogous to the stationary case (3.1). The first of these equations is

$$\begin{aligned}(\Delta^3 - R_c\Delta_2)(\epsilon\tilde{v}^{(20)} + \gamma\tilde{v}^{(11)}) = \epsilon(\delta_j\tilde{v}^{(10)}\partial_j\Delta^2 v^{(10)} + \delta_j v^{(10)}\partial_j\Delta^2\tilde{v}^{(10)}\\ + \sigma^{(10)}\Delta^2\tilde{v}^{(10)}) - \gamma(\Delta z\Delta^2\tilde{v}^{(10)} - \sigma^{(01)}\Delta_2\tilde{v}^{(10)}).\end{aligned} \tag{4.4}$$

By multiplying this equation with the set of function v_l^* and averaging, we obtain a set of equations for the coefficients $\tilde{c}_n$ which represents the solvability conditions for (4.4). We have to conclude that

$$\sigma^{(10)} = \sigma^{(01)} = 0, \tag{4.5}$$

since all other terms vanish identically because of the symmetry of the boundary conditions as in the corresponding case of the stationary equations. Hence (4.4) is solvable for an arbitrary choice of $\tilde{c}_n$. Because of the symmetry of (4.4) in $\tilde{v}^{(10)}$ and $v^{(10)}$ the solutions $\tilde{v}^{(20)}$ and $\tilde{v}^{(11)}$ are given by

$$\left.\begin{aligned} \tilde{v}^{(20)} &= 2\sum_{n,m} \tilde{c}_n c_m \exp\{i(\mathbf{k}_n + \mathbf{k}_m).\mathbf{r}\} F(\mathbf{k}_n.\mathbf{k}_m, z),\\ \tilde{v}^{(11)} &= \sum_n \tilde{c}_n \exp\{i\mathbf{k}_n.\mathbf{r}\} G(z). \end{aligned}\right\} \tag{4.6}$$

Using these solutions we obtain the solvability condition of the equation for $\epsilon^2 v^{(30)} + \epsilon\gamma v^{(21)} + \gamma^2 v^{(12)}$ by multiplying it with the functions v_l^* and averaging. The following set of equations for the coefficients $\tilde{c}_n$ results:

$$\begin{aligned} 0 = \epsilon^2 \sum_n \{(2 - 2\delta_{ln} - \delta_{-ln}) L(\phi_{ln}) + L_1(\phi_{ln})\}(c_n c_{-n}\tilde{c}_l + c_n\tilde{c}_{-n}c_l + \tilde{c}_n c_{-n} c_l)\\ + 2\epsilon\gamma \sum_{nm} B_0 c_m \tilde{c}_n \delta(-\mathbf{k}_l + \mathbf{k}_m + \mathbf{k}_n) + (\gamma^2 D - \epsilon^2 R^{(20)} - \epsilon\gamma R^{(11)} - \gamma^2 R^{(02)})\tilde{c}_l\\ + (\epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)} + \gamma^2\sigma^{(02)}) M\tilde{c}, \end{aligned} \tag{4.7}$$

where $M \equiv \langle f^*(\partial_{zz}^2 - a_c^2)^2 f\rangle$. The terms proportional to ϵ^2 originally are given by:

$$\epsilon^2\left(\sum_{k,n,m}\{A(\mathbf{k}_k.\mathbf{k}_m, \mathbf{k}_m.\mathbf{k}_n, \mathbf{k}_n.\mathbf{k}_k) + 2A(\mathbf{k}_n.\mathbf{k}_m, \mathbf{k}_m.\mathbf{k}_k, \mathbf{k}_k.\mathbf{k}_n)\} \times c_m c_k \tilde{c}_n.\delta(-\mathbf{k}_l + \mathbf{k}_n + \mathbf{k}_m + \mathbf{k}_k) + (\sigma^{(20)}M - R^{(20)})c_l\right) \quad (4.8)$$

and have been rewritten by using the definitions (3.7). Since $R^{(02)}$ is equal to D according to solvability condition (3.8), $\sigma^{(02)}$ has to vanish. The system (4.7) of linear equations for the variables $\tilde{c}_l$ has a solution if, and only if, the characteristic equation

$$\det|a_{ln} + (\epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)})M\delta_{ln}| = 0 \quad (4.9)$$

is satisfied. $\{a_{ln}\}$ is the hermitic matrix of the coefficients of $\tilde{c}_n$ in the first two lines of expression (4.7). The values of $\sigma^{(20)}$ and $\sigma^{(11)}$ for which relation (4.9) holds determine the growth rate

$$\sigma \approx \epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)}, \quad (4.10)$$

since all lower order contributions vanish. The stationary solution v, θ is stable if all possible values of σ are less than zero. In other words, the stationary solution is stable if the matrix $\{a_{ln}\}$ is positive definite.

Comparison of expression (4.7) with the equations (3.8), and the fact that the right side of the same equations can be written as derivatives of the function E', proves that the coefficients a_{ln} are equal to the second derivatives:

$$a_{ln} = \frac{\partial^2 E'}{\partial c_{-l}\partial c_n} \equiv \frac{\partial^2}{\partial c_{-l}\partial c_n}\left\{E - (\epsilon^2 R^{(20)} + \epsilon\gamma R^{(11)} + \gamma^2 R^{(02)})\left(\sum_m c_m c_{-m} - 1\right)\right\}. \quad (4.11)$$

Since a sufficient condition for the minimum of a function is that its first derivatives vanish and the matrix of the second derivatives is positive definite, we conclude: *the function E assumes a minimum value under the side conditions of constant amplitude for those functions among the class* (2.12) *which are stable stationary solutions of the equations* (2.8) *in the limit of small* ϵ *and* γ.

Among the disturbances considered there always exists one solution

$$\tilde{c}_n = c_n, \quad (4.12)$$

which does not satisfy the condition

$$\Sigma(c_n + \tilde{c}_n)(\bar{c}_n + \bar{\tilde{c}}_n) = 1, \quad (4.13)$$

assumed in the formulation of the minimum principle. Since the disturbances can be represented as an orthogonal set, (4.12) is the only disturbance which does not satisfy (4.13). For this disturbance the relation (4.11) yields

$$\epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)} = 2\epsilon^2 R^{(20)} + \epsilon\gamma R^{(11)}.$$

Hence in addition to the minimum of the function E the condition

$$\frac{d[R(\epsilon) - R_c]}{d\epsilon} \geqslant 0 \quad (4.14)$$

is necessary for the stability of the stationary solution. The minimum property, however, together with condition (4.14), is also sufficient for the stability of the stationary solution when ϵ is small enough. To prove this, we show in the appendix that it is sufficient to restrict the stability analysis to the special class of disturbances with $\sigma^{(00)} = 0$, for which the minimum principle has been derived.

The definition of the stability used in the preceding analysis has to be relaxed with respect to disturbances of the special form:

$$\tilde{v} = \tau_j \partial_j v, \quad \tilde{\theta} = \tau_j \partial_j \theta, \tag{4.15}$$

where τ_i is a unit vector with arbitrary horizontal direction. These disturbances are exact solutions of (4.1) with $\sigma = 0$ and correspond to infinitesimal translations of the stationary solution v, θ. Because of the homogeneity of the problem with respect to the horizontal directions, all solutions produced by translations of a given solution,

$$c'_n = c_n \exp\{i\mathbf{k}\,.\,\boldsymbol{\tau}b\}, \tag{4.16}$$

with arbitrary constant b, have identical properties. Neutral disturbances of the form (4.15) are the consequence. The minimum principle does not distinguish between solutions of the class described by (4.16), and it is appropriate to consider all solutions of this class as representing a single solution.

5. Generalization of the minimum principle

In the last sections we have demonstrated the minimum property of stable stationary cellular motion, using the problem (2.2) as a model. The conclusions, however, cover a much wider class of problems, since only the following general properties have been used.

(i) The problem is homogeneous with respect to horizontal directions and the time. Hence the equations are invariant against translations and rotations within the horizontal plane.

(ii) The linear part of the equation is separable with (2.10) governing the horizontal dependence. The solution of the linear problem therefore can be written in the form (2.12, 13). There exists a minimum value of the Rayleigh number for which stationary solutions exist and below which all solutions are exponentially decaying.

(iii) For sufficiently small amplitude, the solution of the problem can be represented by an expansion in powers of the amplitude starting with a solution of the form (2.12) for the linear problem. With the use of this representation any term in the equations can be written according to the properties (i) and (ii) as a sum of terms

$$f_{k, \ldots, n}(z, \mathbf{k}_k, \ldots, \mathbf{k}_n)\, c_k \ldots c_n \exp\{i(\mathbf{k}_k + \ldots + \mathbf{k}_n)\,.\,\mathbf{r}\}.$$

(iv) The functions $f_{k, \ldots, n}$ depend only on the scalar products between the $\mathbf{k}$-vectors, at least up to terms of the order considered.

In the problem (2.2) the property (iv) is due to the vanishing of the vertical component of the vorticity. We have proved this fact restricting ourselves to the case of infinite Prandtl number. It has been shown in I, however, that the vertical component of the vorticity is of the order ϵ^3 or less. Hence the conclusions in the foregoing sections remain valid in the case of finite Prandtl number.

The symmetry of the boundary conditions is not essential to the problem and has been introduced only to simplify the discussion. With the use of different arguments the same results (3.3) can be obtained in cases with non-symmetric boundary conditions. An example is given in § 6.

In the second part of this paper we shall take into account the temperature dependence of any material property. It will be shown that terms similar to the terms proportional to γ in (2.2) are introduced. Hence the conclusions about the minimum property hold in this case as well as for the problem (2.2). In contrast to the expansion in ϵ the expansion in γ is not essential for the conclusions and was introduced only to simplify the analysis. Since no explicit use of the homogeneous boundary conditions has yet been made, the conclusions also cover problems with finitely conducting boundaries, the action of surface tension, or any other boundary conditions as far as they correspond to the properties (i)–(iv).

Many interesting extensions of the convection problem have been discussed, e.g. the convection in a rotating system, the influence of a magnetic field, and the effect of diffusion due to a concentration gradient between the boundaries of the layer. Nonlinear aspects of these problems have been considered by Malkus (1959), Veronis (1959, 1965) and Sani (1965). The diffusion of a substance in general introduces buoyancy forces similar to those due to temperature. Since the concentration is governed by an equation of the same form as the heat equation, the properties (i)–(iv) and hence the minimum principle hold in this case. The same argument is valid in the case of convection with an applied homogeneous vertical magnetic field. The vertical component of the curl of the Lorentz force, as well as the vertical component of the current density, is of the order ϵ^3 and can be neglected.

The minimum principle is not valid, however, for convection in a rotating system because the vertical component of the vorticity is of the order ϵ. The functions in the solvability conditions (3.6, 4.7) depend on $\boldsymbol{\lambda}.(\mathbf{k}_n \times \mathbf{k}_l)$ as well as on $\mathbf{k}_n . \mathbf{k}_l$ in this case. It has been shown by Küppers (1966) that only the regular solutions among the class of semi-regular solutions are possible stationary solutions in this case.

Pellew & Southwell (1940) noticed that the solution of the problem (2.9) can be expressed in terms of a variational principle. The Rayleigh number $R(a)$ in this linear problem can be calculated as the minimum of the functional

$$R(v^{(10)}) \equiv -\frac{\langle(\Delta^3 v^{(10)})\,\Delta^2 v^{(10)}\rangle}{a^2\langle(\Delta v^{(10)})^2\rangle}, \tag{5.1}$$

where $v^{(10)}$ is an arbitrary function of the form (2.12), satisfying the relation (2.10) and the boundary conditions. $R^{(00)}$ is given by the absolute minimum of the functional (5.1). Chandrasekhar (1961) has used this property of the Rayleigh number in more general linear formulations of convection problems.

In analogy to the variational principle for the linear problem, a functional corresponding to the Rayleigh number can be formulated for the nonlinear problem in the approximation to which the equations have been considered:

$$\begin{aligned} R(\epsilon, v^{(10)}) \equiv R(v^{(10)}) &+ \epsilon^2 \sum_{n,l} \{L(\phi_{ln})(2-2\delta_{ln}-\delta_{-ln}) + L_1(\phi_{ln})\} c_n c_{-n} c_l c_{-l} \\ &+ \epsilon\gamma B_0 \sum_{n,m,l} c_l c_m c_n \delta(\mathbf{k}_l+\mathbf{k}_m+\mathbf{k}_n) + \gamma^2 D \sum_l c_l c_{-l}. \end{aligned} \tag{5.2}$$

This functional is defined for all functions $v^{(10)}$ of the form (2.12) which satisfy the boundary conditions. For the present purpose we assume that the value a_c^2 in the relation (2.14) is replaced by the unspecified parameter a^2 and that the functions on the right side of (5.2) are functions of a^2, since their definition was independent of the special value a_c^2. In the special case when $v^{(10)}$ satisfies (2.9) with R equal to $R^{(00)}$, the relation

$$E(\ldots, c_{-1}c_1, \ldots) \equiv \int_0^{\epsilon} R(\epsilon', v^{(10)}) - R(v^{(10)})\,\epsilon'\,d\epsilon' \tag{5.3}$$

holds, and $R(\epsilon, v^{(10)})$ is equal to the Rayleigh number for those solutions $v^{(10)}$ which satisfy the solvability condition. Using this fact we can reformulate the minimum principle.

An arbitrary function of the form (2.12, 13) *corresponds to a stable stationary solution of the convection problem if, and only if, the integral*

$$\int_0^{\epsilon_0} R(\epsilon, v^{(10)})\,\epsilon\,d\epsilon$$

assumes a minimum and $\quad \left.\dfrac{\partial R(\epsilon, v^{(10)})}{\partial \epsilon}\right|_{\epsilon_0}$

is positive, where ϵ_0^2 *is a sufficiently small constant equal to the convective heat flux.*

In the case when γB_0 vanishes, the minimum principle is equivalent to Malkus's hypothesis of maximum heat transport at a given Rayleigh number mentioned in the introduction. Since the minimum principle holds to the order ϵ^2 only, any physical quantity, described as an average property of the stationary solution, can be used as a physical interpretation of the parameter ϵ^2 as, for example, the kinetic energy of convection. The convective heat flux has been used since it corresponds to an experimentally measured quantity.

6. Convection with temperature dependent material properties

Without the Boussinesq approximation the description of convection in a fluid layer heated from below has to start with the general form of the Navier–Stokes equations:

$$\frac{\partial}{\partial t}\rho u_i + u_j \partial_j \rho u_i = -\partial_i p - \rho g \lambda_i + \partial_j[\nu\rho(\partial_i u_j + \partial_j u_i) + \bar{\nu}\rho\delta_{ij}\partial_k u_k], \tag{6.1}$$

the continuity equation
$$\frac{\partial}{\partial t}\rho + \partial_j \rho u_j = 0, \tag{6.2}$$

and the heat equation
$$\frac{\partial}{\partial t}T + u_j\partial_j T = \frac{1}{\rho c_p}\partial_j \lambda\, \partial_j T. \tag{6.3}$$

In writing the last equation we have introduced the assumption that changes in the energy due to pressure fluctuations are small compared with changes due to temperature fluctuations. Representing the fluctuations by the static differences of pressure and temperature between the boundaries we can formulate this assumption as the limit

$$\frac{\alpha\rho g(T_1 - T_2)\,d}{\rho c_p(T_1 - T_2)} = \frac{\alpha g d}{c_p} \to 0. \tag{6.4}$$

This limit also permits one to neglect the dissipation as a heat source in (6.3).

The equations (6.1)–(6.3) give a complete description if an equation of state is added. In accordance with the limit (6.4) we neglect the pressure dependence of the material properties in comparison with the temperature dependence and assume in addition to (2.1)

$$\left.\begin{aligned} \nu &= \nu_0\left(1+\frac{\gamma_2}{T_1-T_2}(T-T_0)+\dots\right), \\ \lambda &= \lambda_0\left(1+\frac{\gamma_3}{T_1-T_2}(T-T_0)+\dots\right), \\ c_p &= c_0\left(1+\frac{\gamma_4}{T_1-T_2}(T-T_0)+\dots\right). \end{aligned}\right\} \tag{6.5}$$

The volume viscosity $\bar{\nu}$ does not enter the problem because the term with $\bar{\nu}$ can be included in the pressure term.

Because of the temperature dependence of the conductivity λ, the solution for the static temperature distribution T_s deviates from a linear dependence in z:

$$T_s - \frac{T_1+T_2}{2} = (T_1-T_2)\left(z-\frac{\gamma_3}{2}\left(z^2-\frac{d^2}{4}\right)\right)+\dots. \tag{6.6}$$

In the same way as in §2 we introduce non-dimensional variables and use the deviation θ from the static temperature distribution T_s in place of T. It is convenient to formulate the problem in terms of the dimensionless momentum vector $v_i = (\rho/\rho_0)u_i$. Thus (6.1)–(6.3) assume the following form in the stationary case:

$$\left.\begin{aligned} \Delta v_i + \theta\lambda_i - \partial_i p &= Pr^{-1} v_j\partial_j v_i + \gamma_0\partial_z v_i - \frac{\gamma_0}{R}(v_j\partial_j\partial_i\theta + \partial_j(v_i\partial_j\theta)) \\ &\quad - Pr^{-1}\gamma_0((z-z_0)v_j\partial_j v_i + v_z v_i) + \gamma_1 2(z-z_0)\theta\lambda_i - \frac{\gamma_1}{R}\theta^2\lambda_i \\ &\quad + \gamma_2(z-z_0)\Delta v_i - \frac{\gamma_2}{R}(\theta\Delta v_i + \partial_j\theta(\partial_i v_j + \partial_j v_i)) + \gamma_2(\partial_z v_i + \partial_i v_z) + \dots, \\ \partial_j v_j &= 0, \\ \Delta\theta + Rv_j\lambda_j &= v_j\partial_j\theta + \gamma_3(z-z_0)\Delta\theta - \frac{\gamma_3}{R}(\theta\Delta\theta + \partial_j\theta\partial_j\theta) + \gamma_3 2\partial_z\theta \\ &\quad + (\gamma_4-\gamma_3)R(z-z_0)v_z - \gamma_4((z-z_0)v_j\partial_j\theta + v_z\theta) + \dots. \end{aligned}\right\} \tag{6.7}$$

In order to simplify the notation we have introduced

$$\gamma_0 = \alpha(T_1-T_2) \quad \text{and} \quad \gamma_1 \equiv \gamma = (\beta/\alpha)(T_1-T_2).$$

Terms of the order $\gamma_\nu\gamma_\mu$ as well as the quadratic temperature dependence of the material properties (6.5) have not been included explicitly in the equations (6.7). The solvability condition (3.6) shows that those terms produce a slight change in the critical Rayleigh number R_c. They do not affect, however, the finite amplitude behaviour of the stationary solution and its stability to the order in which the equations will be considered. Since the momentum vector has been introduced a reformulation of the boundary conditions (2.4) is necessary. Using the continuity equation we obtain

$$\left.\begin{aligned} &v_z = \partial_z v_z = \theta = 0 \text{ at the rigid boundary,}\\ &v_z = \partial_z\{v_{x,y}/(1+\gamma_0 z - \gamma_0\theta/R)\} = \theta = 0 \text{ at the free boundary.}\end{aligned}\right\} \quad (6.8)$$

There are four combinations of these conditions possible at the boundaries $z = \pm\frac{1}{2}$. We call these cases 'rigid–rigid', 'free–free', 'rigid–free', and 'free–rigid'.

The solution of (6.7), together with the four cases of boundary conditions, proceeds in exact analogy to the problem treated in the previous sections. The expansion method differs only in the respect that γ is replaced by the set $\gamma_0, \ldots, \gamma_4$. In place of (2.11) we assume

$$v = \sum_{\mu=1}^{\infty} \epsilon^{\mu} v^{(\mu 0)} + \sum_{\mu=1}^{\infty}\sum_{\kappa=0}^{4} \epsilon^{\mu}\gamma_{\kappa} v_{\kappa}^{(\mu 1)} + \ldots \quad (6.9)$$

and corresponding expressions for the other variables. The z-dependence $f(z)$ of the solution $v^{(10)}$ has been calculated explicitly by Reid & Harris (1958) for all four cases of boundary conditions. The function $f_{fr}(z)$ in the case 'free–rigid' is given, of course, in terms of the function in the case 'rigid–free': $f_{fr}(z) = f_{rf}(-z)$.

In writing (6.7) we have assumed

$$T_0 = T_s(z_0) \quad (6.10)$$

leaving z_0 undetermined. It is appropriate to choose the reference temperature T_0 in such a way that the critical Rayleigh number does not depend upon the parameter γ_κ to the first order. In the case of symmetric boundary conditions, $z_0 = 0$ follows in accordance with the formulation of the problem (2.2). In the case of non-symmetric boundary conditions a non-vanishing co-ordinate z_0 will be determined in general. In I it has been shown that $R^{(10)}$ vanishes for all four cases of boundary conditions. Hence we obtain the result that the first of the inhomogeneous equations analogous to (3.1) introduced by the expansion (6.9) is solvable for any solution $v^{(10)}$ of the form (2.12) with

$$R^{(10)} = \sum_{\kappa=0}^{4} \gamma_\kappa R_\kappa^{(01)} = 0. \quad (6.11)$$

The solution $v^{(20)}$ can be written in the form (3.4) and for $v_\kappa^{(11)}$ expressions analogous to (3.5) are valid. By introducing those solutions into the inhomogeneous terms of the equations (6.7) in the following order the first quantitative result about R as a function of the stationary solution can be obtained.

In the same way the analogy to the problem (2.2) holds for the stability analysis. The equations for the disturbances follow from (6.7) by replacing v and θ by $v+\tilde{v}$ and $\theta+\tilde{\theta}$, taking the part linear in $\tilde{v}, \tilde{\theta}$ and adding $Pr^{-1}\sigma\tilde{v}$ and $\sigma\tilde{\theta}$ respectively on the right side of the resulting equations. Since we are interested in the value of σ only to the first non-vanishing order, it is justified to neglect the term $\partial\rho/\partial t$ in the continuity equation and to assume for $\tilde{v}, \tilde{\theta}$ the same boundary conditions (6.8) as for the stationary solution. With the introduction of the expansion

$$\tilde{v} = \sum_{\mu=1}^{\infty} \epsilon^{\mu-1} \tilde{v}^{(\mu 0)} + \sum_{\mu=1}^{\infty}\sum_{\kappa=0}^{4} \epsilon^{\mu-1}\gamma_\kappa \tilde{v}_\kappa^{(\mu 1)} + \ldots \quad (6.12)$$

and corresponding expressions for $\tilde{\theta}$ and σ the stability analysis proceeds as outlined in §4. The only difference is of quantitative nature because terms with γ_κ occur in the solvability condition (4.7) in place of terms with γ.

Since all qualitative features of the problem can be discussed by means of the special case (2.2) we shall not write down the extensive equations and solvability conditions induced by the expansions (6.9) and (6.12). A detailed explicit description is contained in II. In the following section we shall derive explicit conclusions from (3.8) and (4.7) regarding γB_0 as representative of a linear function P of the parameter γ_κ which will be given in the next section.

7. The hexagon solution

In §3 we have described the special class of semi-regular stationary solutions excepting the case in which an angle of 60° exists between any two **k**-vectors. In general, the solvability condition (3.8) reduces with $R^{(11)} = 0$ to the case for which $\gamma = 0$ unless two **k**-vectors $\mathbf{k}_n, \mathbf{k}_m$ occur with $|\mathbf{k}_n + \mathbf{k}_m| = a_c$ and non-vanishing coefficients c_n, c_m. The terms proportional to γ^2 can be disregarded in this respect since they cancel with $R^{(02)} = D$ identically.

When two such **k**-vectors do occur, a non-vanishing coefficient c_k has to correspond to the vector $\mathbf{k}_k = -\mathbf{k}_n - \mathbf{k}_m$, and all three coefficients c_n, c_m, c_k must have equal absolute value. These conditions and the further restriction

$$c_n c_m c_k = \bar{c}_n \bar{c}_m \bar{c}_k \tag{7.1}$$

follow from (3.8) for $l = n, m, k$. Without losing generality we can assume that two of the coefficients, say c_n and c_m, are real and positive, since this result always can be obtained by a translation of the origin $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0$:

$$c'_n = c_n \exp\{i\mathbf{k}_n . \mathbf{r}_0\}, \quad c'_m = c_m \exp\{i\mathbf{k}_m . \mathbf{r}_0\}. \tag{7.2}$$

According to condition (7.1) the third coefficient has to be real as well. The two possible signs of the term (7.1) correspond to two different solutions, since (7.1) is invariant against translations.

The simplest configuration with an angle of 60° between two **k**-vectors leads to the hexagon solutions

$$v_H^{(10)} = \pm \sum_{\substack{n=-3 \\ n \neq 0}}^{3} \frac{1}{\sqrt{6}} \exp\{i\mathbf{k}_n . \mathbf{r}\} f(z) \quad \text{with} \quad \sum_{n=1}^{3} \mathbf{k}_n = 0, \tag{7.3}$$

which describe convective motions in the form of periodic hexagonal cells. Assuming that $f(z)$ has a positive sign we replace the sign $\pm$ in (7.3) by the symbol s_H which depends on the direction of the motion in the centre of the cell:

$$s_H = \begin{cases} + \text{ when the motion in the centre is upward,} \\ - \text{ when the motion in the centre is downward.} \end{cases} \tag{7.4}$$

The general solution with non-vanishing $R^{(11)}$ can be written as superposition of solutions which result from solutions (7.3) by rotation and translation. We shall, however, restrict the further discussion to the hexagon solutions since it can be shown—a detailed analysis is contained in II—that the superpositions are unstable.

We close the discussion of the solvability condition for the stationary solution by presenting numerical values for the function $L(\phi)$ in table 1 which now includes the terms proportional to the inverse of the Prandtl number. $L_1(\phi)$ is a constant:

$$L_1(\phi) = L(-1). \tag{7.5}$$

The values for the cases with symmetric boundary conditions have been taken from I. In the 'free–rigid', 'rigid–free' cases respectively, the terms depending on the inverse of the Prandtl number have been neglected, since it can be anticipated that they are of the same order of magnitude as in the symmetric cases. This simplification also has been used in the calculation of the values of $R_\kappa^{(11)}$ in the case of the hexagon solutions,

$$R_\kappa^{(11)} = s_H P_\kappa, \tag{7.6}$$

which are given in table 2. The terms proportional to the inverse of the Prandtl number for cases with a rigid boundary will be approximately equal to those in the 'free–free' case. It is worth mentioning that the functions $v_\kappa^{(11)}$ do not have to be computed for the calculation of $R_\kappa^{(11)}$ because the integrals involving $v_\kappa^{(11)}$ can be transformed by partial integration into integrals containing $v^{(20)}$ and γ_κ instead.

	'Free–free'	'Free–rigid' and 'rigid–free'	'Rigid–rigid'
$L(-1)$	$0{\cdot}5$ $+0{\cdot}0Pr^{-1}$ $+0{\cdot}0Pr^{-2}$	0·60167	$0{\cdot}69203$ $+0{\cdot}0Pr^{-1}$ $+0{\cdot}0Pr^{-2}$
$L(-\frac{1}{2})$	$0{\cdot}2596$ $+0{\cdot}08654Pr^{-1}$ $+0{\cdot}05769Pr^{-2}$	0·28774	$0{\cdot}29381$ $+0{\cdot}06842Pr^{-1}$ $+0{\cdot}06003Pr^{-2}$
$L(0)$	$0{\cdot}10571$ $+0{\cdot}06343Pr^{-1}$ $+0{\cdot}03806Pr^{-2}$	0·10838	$0{\cdot}08317$ $+0{\cdot}04232Pr^{-1}$ $+0{\cdot}05947Pr^{-2}$
$L(\frac{1}{2})$	$0{\cdot}0242$ $+0{\cdot}0198Pr^{-1}$ $+0{\cdot}0108Pr^{-2}$	0·03051	$0{\cdot}00486$ $+0{\cdot}00833Pr^{-1}$ $+0{\cdot}03762Pr^{-2}$
$L(1)$	0	0·02551	$0{\cdot}01479$ $-0{\cdot}00944Pr^{-1}$ $+0{\cdot}01665Pr^{-2}$

TABLE 1. The function $L(\phi)$ for different boundary conditions

	'Free–free'	'Free–rigid' and 'rigid–free'	'Rigid–rigid'
P_0	$1{\cdot}591\ -0{\cdot}1258Pr^{-1}$	2·142	2·676
P_1	$-4{\cdot}522\ -0{\cdot}5023Pr^{-1}$	−5.651	−6·603
P_2	$2{\cdot}177\ +0{\cdot}0Pr^{-1}$	2·452	2·755
P_3	$2{\cdot}010\ -0{\cdot}5023Pr^{-1}$	2·416	2·917
P_4	$-4{\cdot}271\ +0{\cdot}2512Pr^{-1}$	−5·190	−6·229

TABLE 2. The values P_κ

Since the equations (2.8) for the model case discussed in §§3 and 4 follow from (6.7) with $Pr = \infty$, $\gamma_0 = \gamma_2 = \gamma_3 = \gamma_4 = 0$, the value B_0 is given by

$$B_0 = (3/2)^{\frac{1}{2}} P_1.$$

In general γB_0 has to be replaced by a linear function P of the parameter γ_κ.

It is convenient to choose P equal to the sum of the terms $\gamma_\kappa P_\kappa$. Hence, the replacement is formally expressed by

$$\gamma B_0 \to (3/2)^{\frac{1}{2}} P \equiv (3/2)^{\frac{1}{2}} \sum_{\kappa=0}^{4} \gamma_\kappa P_\kappa. \tag{7.7}$$

The dependence of the Rayleigh number on the amplitude to the order which has been considered is given by

$$R - R_c = \epsilon^2 R^{(20)} + \epsilon \sum_{\kappa=0}^{4} \gamma_\kappa R_\kappa^{(11)}. \tag{7.8}$$

Since the values of $R^{(20)}$ are always positive, the hexagon solutions have the special property that values of R below the critical value R_c depending on the sign of P are possible. The stability analysis in the next section will give the result that $s_H P$ has to be negative for the stable hexagon solution.

In addition to the values of $R^{(20)}$ given in I we note the values for the case of mixed boundaries with infinite Prandtl number:

rolls: $c_1 \bar{c}_1 = \frac{1}{2}$,
$R_R^{(20)} = 0{\cdot}61442$;

square cells: $c_1 \bar{c}_1 = c_2 \bar{c}_2 = \frac{1}{4}$,
$R_S^{(20)} = 0{\cdot}71643$;

hexagons: $c_1 \bar{c}_1 = c_2 \bar{c}_2 = c_3 \bar{c}_3 = \frac{1}{6}$,
$R_H^{(20)} = 0{\cdot}81809$.

Since the convective heat transport $H = \langle u_z \theta \rangle$ is given by ϵ^2 according to the normalization (2.17) the dependence of the heat flux on the Rayleigh number can be derived easily from (7.8).

8. The exchange of stability between hexagons and rolls

It has been shown in I that for small amplitudes ϵ the only stable solution of (6.7) in the case when all γ_κ vanish is the two-dimensional solution which corresponds to a convective motion in the form of rolls. In this section we investigate how this result is modified when the temperature dependence of the material properties is included. We base the discussion on the equations in § 4 regarding γ as representative for the set γ_κ and use the function P defined in (7.7) in the explicit results.

We consider first the class of stationary solutions for which the coefficients c_n or c_m vanish whenever $|\mathbf{k}_n + \mathbf{k}_m| = a_c$. It can be shown that those solutions, including the two-dimensional solution, are unstable for sufficiently small amplitudes. Let us consider the following special disturbance of the form (4.3)

$$\tilde{v}^{(10)} = \tilde{c}_\rho v_\rho + \tilde{c}_{-\sigma} v_{-\sigma} \quad \text{with} \quad \mathbf{k}_\rho + \mathbf{k}_\sigma + \mathbf{k}_m = 0, \tag{8.1}$$

where $\mathbf{k}_m$ is an arbitrary $\mathbf{k}$-vector corresponding to a non-vanishing coefficient c_m of the stationary solution. A disturbance of the special form (8.1) is possible because $\tilde{c}_\rho$ and $\tilde{c}_{-\sigma}$ multiplied by a non-vanishing term appear only in the equations $l = \rho, -\sigma$ of the general solvability condition (4.7). These two equations have the following form in the case of the two-dimensional solution with $m = 1$

when the expression for $R^{(20)}$ given by equations (3.8) and the relation (7.7) is taken into account:

$$\left.\begin{aligned} 0 &= (M\sigma + L_2\epsilon^2)\tilde{c}_\rho + \sqrt{6}\,\epsilon P\bar{c}_1\tilde{c}_{-\sigma}, \\ 0 &= (M\sigma + L_2\epsilon^2)\tilde{c}_{-\sigma} + \sqrt{6}\,\epsilon P c_1\tilde{c}_\rho. \end{aligned}\right\} \tag{8.2}$$

To simplify the notation we have introduced

$$L_2 \equiv L(\tfrac{1}{2}) + L(-\tfrac{1}{2}) - \tfrac{1}{2}L(1) \tag{8.3}$$

and used the approximate relation

$$\sigma \approx \epsilon^2\sigma^{(20)} + \epsilon \sum_{\kappa=1}^{4} \gamma_\kappa \sigma_\kappa^{(11)} \tag{8.4}$$

since terms of higher order will be neglected.

The solvability condition for the system (8.2) of two linear homogeneous equations determines the two roots

$$\sigma M = -L_2\epsilon^2 \pm \sqrt{3}\,\epsilon P. \tag{8.5}$$

Two additional roots for which the same relation (8.5) holds are obtained when m is chosen equal to -1. Those four roots are the only ones of the general characteristic equation (4.9) in which a term proportional to γ or P respectively appears. Hence the two-dimensional solution is unstable for

$$\epsilon \leqslant \epsilon_R \equiv \sqrt{3}\,|P|/L_2, \tag{8.6}$$

but stable for amplitudes above this value.

The other solutions yield expressions similar to (8.5) for the disturbances of the form (8.1). This does not affect, however, the result of I, that they are unstable anyway as long as there is no angle of 60° between two of their **k**-vectors, because for the class of disturbances considered in I the argument of the δ-function in (4.7) never vanishes.

We turn now to the case when an angle of 60° occurs between two **k**-vectors corresponding to non-vanishing coefficients c_n. In the last section we chose the hexagon solutions as representative of this class of stationary solutions. The characteristic equation (4.9) for these solutions reduces to the following equations, since all matrix elements a_{ln} with $|n| > 3$ or $|l| > 3$ vanish unless $n = l$:

$$\det\begin{vmatrix} \sigma M + e & b & b & b + Ps_H & b + Ps_H & e \\ b & \sigma M + e & b & b + Ps_H & e & b + Ps_H \\ b & b & \sigma M + e & e & b + Ps_H & b + Ps_H \\ b + Ps_H & b + Ps_H & e & \sigma M + e & b & b \\ b + Ps_H & e & b + Ps_H & b & \sigma M + e & b \\ e & b + Ps_H & b + Ps_H & b & b & \sigma M + e \end{vmatrix} = 0 \tag{8.7}$$

and for $|l| > 3$

$$\epsilon^2 \sum_n \{2L(\phi_{ln}) - (2 - 2\delta_{ln} - \delta_{-ln})L(\phi_{ln})\}\,\tfrac{1}{6} - \epsilon Ps_H + M\sigma = 0. \tag{8.8}$$

As abbreviations we have introduced

$$\left.\begin{aligned} e &\equiv \tfrac{1}{3}\epsilon^2(L(-1)+\tfrac{1}{2}L(1)) - \epsilon P s_H, \\ b &\equiv \tfrac{1}{3}\epsilon^2(L(-1)+L(\tfrac{1}{2})+L(-\tfrac{1}{2})). \end{aligned}\right\} \tag{8.9}$$

The following six growth rates are determined as zeros of equation (8.7). The range in which they are positive has been indicated under the assumption that s_H and P have opposite signs:

$$\left.\begin{aligned} \sigma_1 M &= -\epsilon^2 2(L(-1)+\tfrac{2}{3}L(\tfrac{1}{2})+\tfrac{2}{3}L(-\tfrac{1}{2})+\tfrac{1}{6}L(1)) - \epsilon s_H P \\ &= -2\epsilon^2 R_H^{(20)} - \epsilon s_H P \geqslant 0 \quad \text{for} \quad \epsilon \leqslant \epsilon_A \equiv \frac{|P|}{2R_H^{(20)}}, \\ \sigma_2 M &= \sigma_3 M = \frac{2\epsilon^2}{3} L_2 + 2\epsilon s_H P \\ &\geqslant 0 \quad \text{for} \quad \epsilon \geqslant \epsilon_B \equiv \frac{3|P|}{L_2}, \\ \sigma_4 M &= +3\epsilon s_H P, \\ \sigma_5 M &= \sigma_6 M = 0. \end{aligned}\right\} \tag{8.10}$$

The other possibility $s_H P > 0$ can be disregarded since it leads to a solution which always is unstable according to the fourth growth rate in (8.10).

The growth rates given by (8.8) do not lead to instability as long as $s_H P$ is negative. This fact holds even for vanishing P and has been proven in II using the concave dependence of the function $L(\phi)$ on ϕ. Hence the range of stability for the hexagon solutions is determined by ϵ_A and ϵ_B. Unlike the hexagon solution only one form of convection corresponds to the two-dimensional solution since a change of the sign leads to the same but translated solution. Thus there are three possibly stable convective motions, the stability regions of which are indicated in figure 1.

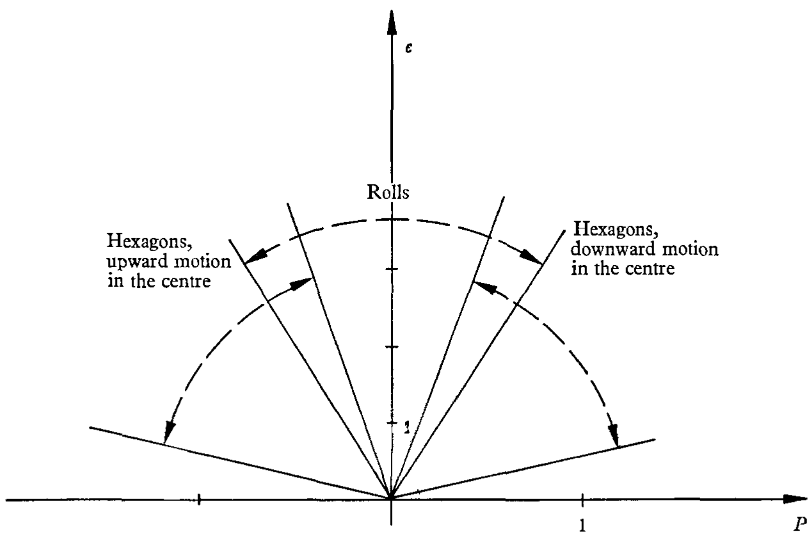

FIGURE 1. Regions of stability for the three possible convective motions. The scale corresponds to the case 'rigid–rigid' with $Pr = \infty$.

A more physical picture is given in figure 2 where the dependence of the amplitude on the Rayleigh number has been plotted qualitatively. The curves are parabolas to the approximation in which the problem has been analysed and the Rayleigh numbers corresponding to ϵ_A, ϵ_B, ϵ_R are given by

$$\left.\begin{aligned} R_A - R_c &= -\frac{P^2}{4R_H^{(20)}}, \\ R_B - R_c &= P^2\frac{9R_H^{(20)} - 3L_2}{L_2^2}, \\ R_R - R_c &= P^2\frac{3R_R^{(20)}}{L_2^2}. \end{aligned}\right\} \tag{8.11}$$

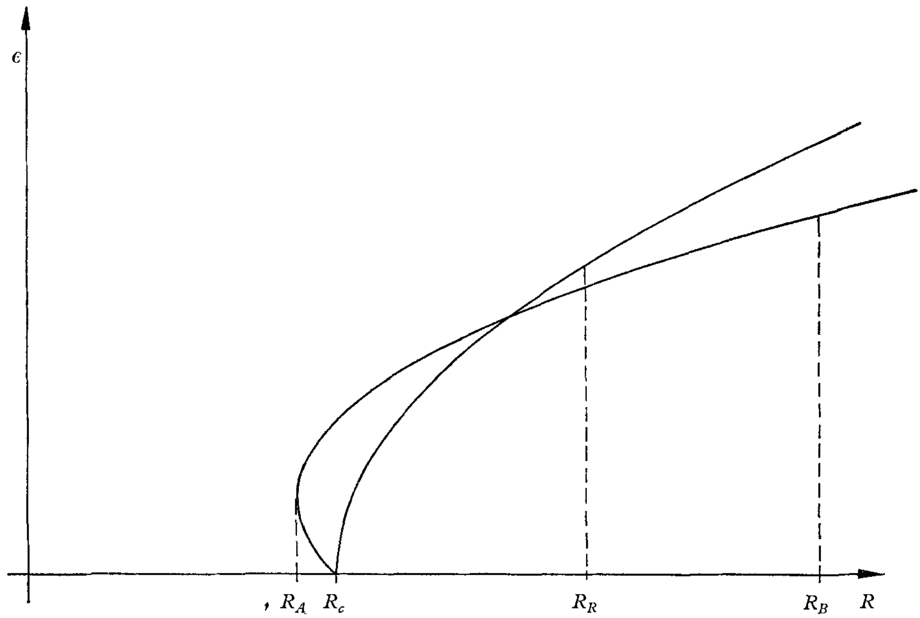

FIGURE 2. Qualitative sketch for the dependence of the amplitude on the Rayleigh number for rolls and hexagons with the boundaries of stability R_A, R_R, R_B.

When the Rayleigh number is slowly increased the convection starts growing at the critical Rayleigh number and settles at the finite amplitude value of the stable stationary hexagon solution. At Rayleigh number R_B, the hexagonal convection pattern becomes unstable. The growing disturbances transform the hexagon solution into rolls, corresponding to one of the three **k**-vectors, $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. This fact follows from the form of the disturbances belonging to the growth rates σ_2 and σ_3 and has been shown in the experiment by Silveston (1958).

With decreasing Rayleigh number the transition from rolls to hexagons occurs at R_R and the convection decays after $R = R_A$ has been passed. The fact that the convection at a certain Rayleigh number depends on the way in which the Rayleigh number has been reached is called the hysteresis effect.

The exchange of stability between the three possible cellular motions, when P is varying at approximately constant Rayleigh number, has been observed by von Tippelskirch (1957). In his experiments with tobacco smoke aerosol, the condensation of water vapour provided an additional heat transport by diffusion with strongly decreasing temperature dependence. Hence hexagons with upward motion in the centre were observed. When the water droplets slowly disappeared the convection turned into the form of rolls and later into hexagons with downward motion in the centre, which is characteristic for gases.

9. Concluding remarks

The stability analysis in the last section has shown that the temperature dependence of all material properties has to be taken into account for the determination of the physically realized convective motion at small amplitudes. Three forms of convection are possibly stable. Two of these correspond to a hexagonal cell pattern and are mirror images of one another, with respect to the middle plane of the layer, when the boundary conditions are symmetric. The mirror image of the third possible solution describing convection in the form of rolls leads to the same but translated solution. These symmetry properties are reflected in the stability analysis. At small amplitudes, when the inhomogeneities of the layer due to the temperature dependence of the material properties become important, the convection in hexagonal cells is preferred. When the conditions are sufficiently symmetric, rolls—the simplest form of convection—are realized. It has been shown in II that non-symmetric boundary conditions can lead to a preference of hexagons at higher amplitudes. Several authors have concluded the preference of the hexagons taking into account the temperature dependence of viscosity only; see Palm & Øiann (1964) and Segel (1965) and their references to earlier work. The stability of the hexagon solutions was first shown in II. Heat sources in the fluid and time dependent temperatures at the boundaries have not been considered in the present work. Krishnamurti (1967) has shown that in these cases also hexagons are preferred.

According to the stability analysis two solutions may be stable at the same given situation and the initial conditions may determine which solution is realized. This hysteresis effect is in agreement with the minimum principle because the function E may have more than one minimum. The absolute minimum, however, together with the condition (4.14) ensures the stability of the corresponding solution.

No stable stationary convection with amplitude less than ϵ_A is possible. This fact raises the question of which kind of convection is realized when the heat flux corresponding to an amplitude in the unstable range is prescribed. The problem has been analysed in another paper, Busse (1967*a*), with the result that convection with periodic time dependence can occur in this situation.

Throughout this work we have assumed that the wave-number a of the stationary solution is equal to a_c which corresponds to the lowest possible Rayleigh number of the linear equations. The discussion in the appendix indicates that there exists for any stable solution a finite region of possible wave-number a in the neighbourhood of a_c. Because the width of this region is of the order ϵ, the corresponding solutions differ only by terms of higher order than those considered in this work. At higher Rayleigh number, however, the dependence of the solutions on the wave-number becomes an interesting problem. In the special case of two-dimensional convection, it has been studied in a recent paper (Busse 1967*b*). We refer to this paper also with respect to the question of how far the approximation used in the present work gives a good representation of the solution. The analysis of that paper shows that the influence of the terms proportional to ϵ^n with $n > 2$ is small unless R exceeds the order of two or three times the critical value. The expansion with respect to the parameter γ_κ is similar—at least for the linear problem—to a method used by Chandrasekhar (1961, pp. 309–13) in the stability analysis of circular Couette flow. Chandrasekhar shows that the second-order results give a remarkably close approximation to the exact solution even for values which correspond to values γ_κ of the order one.

The author is indebted to Professor A. Schlüter who introduced him to the problem of cellular convection.

Appendix

In §4 we have restricted the stability analysis to disturbances with $\sigma^{(00)} = 0$. Disturbances with $\sigma^{(00)} > 0$ cannot exist when the basic wave-number a of the stationary solution is equal to a_c. We must show, however, that disturbances with $\sigma^{(00)} < 0$ do not alter the conclusion of §4 when $|\sigma^{(00)}|$ becomes of the order ϵ^2 or $\epsilon\gamma$. It is convenient to neglect $\sigma^{(00)}$ in this case in the lowest order of (4.1) and to consider it in the same order as $\epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)}$. We recapitulate the stability analysis of §4 using instead of (4.3) the more general expression

$$\tilde{v}^{(10)} = \sum_q \tilde{c}_q \hat{v}_q, \tag{A1}$$

where the set of functions

$$\hat{v}_q = \exp\{i\hat{\mathbf{k}}_q.\mathbf{r}\}\hat{f}(|\hat{\mathbf{k}}_q|, z) \tag{A2}$$

represents the system of solutions of (2.9) for $R = R(|\hat{\mathbf{k}}_q|^2)$. We use the symbol ^ to distinguish the present analysis from the special case treated earlier in which the vectors $\mathbf{k}$ satisfy the second condition in (2.14) in addition to the first. Since $R^{(00)}$ is defined as the minimum of $R(a)$ we can use as an approximation

$$R \approx R^{(00)} + \eta(|\hat{\mathbf{k}}_q|^2 - a_c^2), \tag{A3}$$

with a positive constant η.

In analogy to (2.16) the set of functions

$$\hat{v}_q^* = \exp\{-i\hat{\mathbf{k}}_q.\mathbf{r}\}\hat{f}^*(|\hat{\mathbf{k}}_q|^2, z), \tag{A4}$$

is defined as the system of solutions of the adjoint problem to (2.9).

The analysis of (4.1), in which the expansions (2.11) and (4.2) have been inserted, proceeds in the same manner as in the special case discussed in §4. By multiplying the equation for

$$\tilde{v} = \tilde{v}^{(10)} + \epsilon\tilde{v}^{(20)} + \gamma\tilde{v}^{(11)} + \epsilon^2\tilde{v}^{(30)} + \dots$$

with $\hat{v}_p^*$ and averaging, we obtain the solvability condition represented by a set of homogeneous equations with running index p for the coefficients $\tilde{c}_q$ of $\tilde{v}^{(10)}$.

In the order ϵ, γ of these equations we have to conclude $\sigma^{(10)} = \sigma^{(01)} = 0$ as in (4.5) because of the symmetry of the problem. A non-trivial set of equations similar to (4.7) is obtained in the order ϵ^2, $\epsilon\gamma$, γ^2. As in (4.9) the condition for the solvability of this set of equations can be expressed by

$$\det |\hat{a}_{pq} + [\hat{\sigma}M + \eta(|\hat{\mathbf{k}}_q|^2 - a_c^2)^2]\,\delta_{pq}| = 0, \tag{A 5}$$

$\{\hat{a}_{pq}\}$ is the matrix of the coefficients of $\tilde{c}_q$ corresponding to the matrix $\{a_{ln}\}$. As $\{a_{ln}\}$ it is hermitic and contains only terms of the order ϵ^2, $\epsilon\gamma$. Since $\sigma^{(02)}$ is vanishing as in §4,

$$\hat{\sigma} \equiv \sigma^{(00)} + \epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)} \tag{A 6}$$

is identical with the growth rate σ in the approximation in which the problem will be considered. The term proportional to η in (A 5) represents the fact that we are using the more general set of functions (A 2) rather than (2.13).

For simplicity let us assume that only a finite number, say $2N$, of the coefficients c_n of the stationary solution are different from zero. Let $\{\mathbf{k}_n, |n| \leqslant N\}$ be the set of vectors $\mathbf{k}_n$ corresponding to these coefficients. By adding all vectors $\mathbf{k}_k$ with $\mathbf{k}_n . \mathbf{k}_k = \frac{1}{2}a_c^2$ as far as they do not yet belong to the set, we obtain the enlarged set $\{\mathbf{k}_n, |n| \leqslant N^*\}$ with $N^* \leqslant 3N$. In this notation the matrix elements a_{ln} are vanishing unless $n = l$ or $|n| \leqslant N^*$ and $|l| \leqslant N^*$. Hence (4.9) can be rewritten as

$$\det |a'_{ln} + \tilde{\sigma}M\delta_{ln}| \prod_{|l|>N^*} [a_{ll} + \tilde{\sigma}M] = 0, \tag{A 7}$$

with the submatrix $\{a'_{ln}\}$ of $\{a_{ln}\}$ defined by

$$a'_{ln} = a_{ln} \quad \text{for} \quad |l|, |n| \leqslant N^*. \tag{A 8}$$

$\tilde{\sigma}$ is an abbreviation for $\epsilon^2\sigma^{(20)} + \epsilon\gamma\sigma^{(11)}$. Equation (A 3) can be rewritten in a similar form. Since the non-diagonal elements $\hat{a}_{pq}$ vanish unless the relation

$$-\hat{\mathbf{k}}_p + \hat{\mathbf{k}}_q + \mathbf{k}_n = 0, \quad \text{or} \quad -\hat{\mathbf{k}}_p + \hat{\mathbf{k}}_q + \mathbf{k}_n + \mathbf{k}_k = 0,$$

with $|n|, |k| \leqslant N^*$, is satisfied, the elements $\hat{a}_{pq}$ with $p \neq q$ can be different from zero only if $\hat{\mathbf{k}}_p$ and $\hat{\mathbf{k}}_q$ belong to the same subset of vectors $\hat{\mathbf{k}}$

$$\{\mathbf{k}_n + \mathbf{d}, |n| \leqslant N^*\}, \tag{A 9}$$

where $\mathbf{d}$ is a constant vector. Let us denote the submatrix of $\{\hat{a}_{pq}\}$, where this condition is satisfied, by $\{\hat{a}'_{ln}(\mathbf{d})\}$. Then the left side in (A 3) can be written as a product of terms of the form

$$\det |\hat{a}'_{ln}(\mathbf{d}) + (\hat{\sigma}M + \eta(\mathbf{k}_n . \mathbf{d} + |\mathbf{d}|^2)^2)\,\delta_{ln}|, \tag{A 10}$$

and the remaining diagonal elements

$$\hat{a}_{pp} + \hat{\sigma}M + \eta(|\hat{\mathbf{k}}_p|^2 - a_c^2)^2. \tag{A 11}$$

(A 10) becomes identical to the first term in (A 7) when $\mathbf{d} = 0$. Similarly, there exists for any term (A 11) a corresponding term among the factors in (A 7) to which it reduces when $|\mathbf{d}|, |\hat{\mathbf{k}}_p| - a_c \to 0$. This correspondence shows that the values $\hat{\sigma}$ determined by (A 5) differ from the values $\tilde{\sigma}$ given by (4.8) or (A 7) respectively by terms of the order $\epsilon^2|\mathbf{d}|, \epsilon\gamma|\mathbf{d}|$ and $|\mathbf{d}|^2$. The latter terms are proportional to η and give a negative contribution to the growth rate $\hat{\sigma}$. Hence the values $\hat{\sigma}$ are negative for arbitrary vectors $\mathbf{d}$, when the corresponding value $\tilde{\sigma}$ is negative and ϵ is sufficiently small.

The case $\tilde{\sigma} = 0$ needs special consideration. In §4 it has been shown that there exists for any stable stationary solution a class of disturbances given by

$$\tilde{c}_n = ic_n \boldsymbol{\tau}.\mathbf{k}_n, \tag{A 12}$$

with $\sigma = 0$. Using the property

$$\hat{a}'_{ln}(\mathbf{d}) = \hat{a}'_{-n-l}(-\mathbf{d}), \tag{A 13}$$

of the matrix $\{\hat{a}'_{ln}\}$ we show that the corresponding values $\hat{\sigma}$ are negative. By multiplying the matrices occurring in the expressions (A 7) and (A 10) with the eigenvector (A 12) of a'_{ln} we obtain

$$\begin{aligned} M\hat{\sigma}\sum_n \tilde{c}_n\bar{\tilde{c}}_n &= -\eta \sum_n (|\hat{\mathbf{k}}_n|^2 - a_c^2)^2 \tilde{c}_n \bar{\tilde{c}}_n - \sum_{l,n} (\hat{a}'_{ln} - a'_{ln}) \bar{\tilde{c}}_l \tilde{c}_n \\ &= -\eta \sum_n (2\mathbf{d}.\mathbf{k}_n + |\mathbf{d}|^2)^2 (\boldsymbol{\tau}.\mathbf{k}_n)^2 c_n \bar{c}_n - \sum_{l,n} \mathbf{b}'_{ln}.\mathbf{d}\bar{\tilde{c}}_l \tilde{c}_n + \ldots, \end{aligned} \tag{A 14}$$

where $\mathbf{b}'_{ln}$ is defined as the gradient of a'_{ln} with respect to $\mathbf{d}$ at $\mathbf{d} = 0$. The dots indicate terms proportional to $|\mathbf{d}|^2$ and higher powers of $|\mathbf{d}|$. The summation is extended over the range $0 < |n|, |l| \leqslant N^*$. Any disturbance of the form (A 12) has the property

$$\tilde{c}_n = \bar{\tilde{c}}_{-n}. \tag{A 15}$$

Disturbances corresponding to other growth rates do not have this property in general, although to any growth rate there exists always at least one disturbance satisfying (A 15). Using (A 15) and (A 13) we find

$$\sum_{ln} \mathbf{b}'_{ln}.\mathbf{d}\bar{\tilde{c}}_l\tilde{c}_n = -\sum_{ln} \mathbf{b}'_{-n-1}\bar{\tilde{c}}_l\tilde{c}_n = -\sum_{ln} \mathbf{b}'_{-n-l}\bar{\tilde{c}}_{-n}\tilde{c}_{-l} = 0,$$

and conclude that $\hat{\sigma}$ is negative, since the first term on the right side of (A 14) is of the order $|\mathbf{d}|^2$ and negative and cannot be balanced by the following terms as long as ϵ is sufficiently small.

From the last statement we have to except one single case. For the two-dimensional solution with $N = 1$ the first term on the right side of (A 14) is proportional to $|\mathbf{d}|^4$ when $\mathbf{k}_1 \mathbf{d} = 0$. Hence $\hat{\sigma}$ may become positive due to terms of the order $\epsilon^2|\mathbf{d}|^2$. (Terms of the order $\epsilon\gamma$ do not enter the expression for $\hat{\sigma}$.) In this case, however, the two-dimensional solution with $|\mathbf{k}_1|^2 = a_c^2 + \zeta$, $\zeta > 0$ will be stable provided the two-dimensional solution is stable with respect to the disturbances discussed in §4. The form

$$-\tfrac{1}{2}\eta\{(\zeta + 2\mathbf{k}.\mathbf{d} + |\mathbf{d}|^2)^2 + (\zeta - 2\mathbf{k}_1.\mathbf{d} + |\mathbf{d}|^2)^2 - 2\zeta^2\}$$

of the first term on the right side of (A 14) shows that σ is negative for

$$\zeta > \alpha_1 \epsilon^2. \tag{A 16}$$

The possibly positive contribution of this term in the case of other disturbances does not lead to instability as long as an inequality of the form

$$\zeta^2 < \alpha_2 \epsilon^2, \tag{A 17}$$

is satisfied. Relations (A 16) and (A 17), with appropriately chosen positive constants α_1 and α_2 can be satisfied for sufficiently small ϵ. In order to make the analysis applicable to the more complex cases mentioned in §5 we have used only qualitative arguments. In the special case treated in I the terms of the order $\epsilon^2|\mathbf{d}|^2$ do not lead to instability and the two-dimensional solution is stable for $|\mathbf{k}_1| = a_6$.

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